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Some equivalent formulations of the intersection problem of finitely generated classes of graphs

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Introduction. An ordering $\prec$ on the class of finite undirected graphs without loops is defined by $G \prec F$ iff there exists a (partial) subgraph G of the graph F which is a subdivision of the graph G. A class L of graphs is called *closed* if $G \in L$, $G \prec F \Rightarrow F \in L$. By $L(G_1, \ldots, G_n)$ we denote the smallest class of graphs which is closed and contains the graphs $G_1, \ldots, G_n$. The graphs $G_1, \ldots, G_n$ are called *generators* of the class $L(G_1, \ldots, G_n)$. A class L is called *finitely generated* if it is closed and there are graphs $G_1, \ldots, G_n$ such that $L = L(G_1, \ldots, G_n)$. If L is a closed class we denote by $B(L)$ the set of all minimal members of L in $\prec$. The set $B(L)$ is called the *base* of L. Evidently, L is finitely generated iff its base $B(L)$ is finite.

The following problem was posed by L. LOVÁSZ [1] and by P. UNGAR [3]: Is the class $L \cap L'$ finitely generated for every pair L, L' of finitely generated classes of graphs? It is not difficult to see that the essence of the problem lies in the investigation of "braids" of subdivisions of pairs of graphs. The problem is equivalent to the question whether the number of "critical braids" is finite or infinite.

Our method shows that it is sufficient to investigate such "braids" of subdivisions G', H' of graphs G, H that G' does not contain vertices of H and H' does not contain vertices of G. Every edge of a graph determines a path in its subdivision. If we decompose the graphs G, H into single edges, it is sufficient to investigate the "braids" of corresponding paths. This "braid" of paths will be called a *crossing system* (see the definition below). We hope that the investigation of "braids" of paths is easier than the investigation of "braids" of general graphs and could lead to a solution of the problem. It also follows that the problem does not depend on concrete graphs. We prove that it is sufficient to solve it for special pairs $L(G)$, $L(H)$ where G is a disjoint union of complete graphs K_6^{5} and H is a disjoint union of complete bipartite graphs $K_{2,6}$.

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Notions and results. A graph $c=(\{v_0, \dots, v_t\}, \{e_1, \dots, e_t\})$ is called a *path* if e_i is edge adjacent to vertices v_{i-1}, v_i, $1 \leqq i \leqq t$. Denote by $V(c)$ the set $\{v_0, \dots, v_t\}$ of vertices of the path c, and by $K(c)=\{v_0, v_t\}$ the set of endvertices of the path c. A set of paths $C=\{c_1, \dots, c_m\}$ is called a *disjoint system of paths* if every two paths of C are vertex disjoint. Put $V(C)=\bigcup_{i=1}^{m} V(c_i)$, $K(C)=\bigcup_{i=1}^{m} K(c_i)$. $K(C)$ is called the set of endvertices of C.

Let $C=(c_1, \dots, c_m)$, $D=(d_1, \dots, d_n)$ be two disjoint systems of paths which satisfy $K(C)\cap V(D)=V(C)\cap K(D)=\emptyset$. In this case the couple (C, D) is called an *(m, n)-crossing system*. By $\mathrm{gr}\,(C, D)$ we denote the graph on the set of vertices $V(C)\cup V(D)$ which is the union of all paths of C, D. A vertex $v\in V(C)\cap V(D)$ is called a *crossing* of (C, D) if $N(v, C)\neq N(v, D)$ where $N(v, C)$, resp. $N(v, D)$, is the set of all neighbours of the vertex v in the graph C, resp. D.

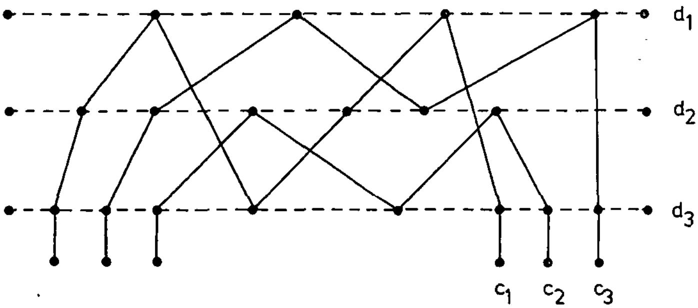

Fig. 1a

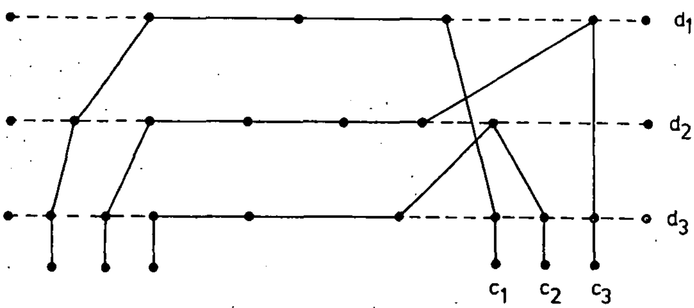

Fig. 1b

Fig. 1a, 1b are examples of (3, 3)-crossing systems. The crossing system 1a is reducible. An example of its reduction is the crossing system in Fig. 1b.

There is a crossing at every vertex in the crossing system in Fig. 1a but not in Fig. 1b.

We say that an (m, n)-crossing system (C, D) is *reducible* if in $\mathrm{gr}(C, D)$ there exist two disjoint systems of paths C', D' such that

1) $C' = (c_1', \ldots, c_m')$, $D' = (d_1', \ldots, d_n')$;

2) $K(c_i) = K(c_i')$, $K(d_j) = K(d_j')$ for every i, j, $i = 1, \ldots, m$, $j = 1, \ldots, n$;

3) the crossing system (C', D') has strictly fewer crossings than (C, D). (See Fig. 1.)

Denote by $G+H$ the disjoint sum of G and H.

Theorem. *The following conjectures are equivalent:*

1) $L_1 \cap L_2$ *is a finitely generated class for every pair* L_1, L_2 *of finitely generated classes of graphs.*

2) *The class* $L(G) \cap L(H)$ *is finitely generated for every two graphs* G, H.

3) *The class* $L(K_6 + \ldots + K_6) \cap L(K_{2,6} + \ldots + K_{2,6})$ *(the graphs in brackets are disjoint unions of* n *copies of* K_6, *resp.* $K_{2,6}$*) is finitely generated for every natural number* n.

4) *For every* m *and* n *there exists a* k *such that every* (m, n)*-crossing system with more than* k *crossings is reducible.*

Proofs. Evidently 1)⇔2) and 2)⇒3). We prove 3)⇒4) and 4)⇒2). The crossing system (C, D) is called *minimal* if (C, D) is not reducible and every vertex of $\mathrm{gr}(C, D)$ is crossing. The implication 4)⇒2) immediately follows from the following lemma.

Lemma 1. *Let a graph* B *belong to the base of the class* $L(G) \cap L(H)$ *and let* m *and* n *denote the numbers of the edges of graphs* G *and* H, *resp. Then there exists a minimal* (m, n)*-crossing system with at least* $|B|$ *vertices.*

Proof. Let a graph R contain subdivisions G', H' of the graphs G, H. We may suppose G, H have no isolated vertices. In general these subdivisions can be placed differently in the graph R. Therefore we introduce the following notation. We denote by $\varphi_G\colon G \to R$ the morphism which maps the graph G on its subdivision $G' = \varphi G$ in the graph R: the morphism φ_G maps the vertices of G on distinct vertices of the graph R and the edges of the graph G on openly disjoint paths. The location of the subdivision of the graph H we denote similarly by $\varphi_H\colon H \to R$. Put $\varphi = (\varphi_G, \varphi_H)$. In the sequel a *morphism* will always mean such a pair $\varphi = (\varphi_G, \varphi_H)$. Every morphism $\varphi = (\varphi_G, \varphi_H)$ induces a *vertex-mapping* $f_\varphi\colon V(G+H) \to V(R)$ which is the restriction of the morphism φ to the set of vertices of $G+H$. Clearly a vertex-mapping $f\colon V(G+H) \to V(R)$ can be induced by various morphisms. If $e = (a_1, a_2)$ is an edge of G then the image of the edge e is a path $\varphi(e) = \varphi(a_1, a_2) = (f(a_1) = x_0, x_1, \ldots$ $\ldots, x_{k-1}, x_k = f(a_2))$, $x_i \in V(\varphi G)$, $k \geqq 1$. A vertex $a \in V(G)$ is called a *tied vertex* in R with respect to φ if $f(a)\in\varphi H$. Likewise $b\in V(H)$ is a *tied vertex* in R with respect to φ if $f(b)\in\varphi G$. The set of tied vertices of the graphs G, H is denoted by $W\varphi=W$. (So $W\subseteqq V(G)\cup V(H)$.)

We shall study quadruples (R, φ, f, W) where R is a graph, φ is a morphism, f is a vertex-mapping and W is a set of tied vertices. The quadruple (R, φ, f, W) is *admissible* if $\varphi=(\varphi_G, \varphi_H)\colon G+H\to R$ and $f=f_\varphi$ is the vertex mapping induced by φ and $W=W\varphi$ is the set of tied vertices with respect to φ. The admissible quadruple (R, φ, f, W) is called *critical* if:

1) after removing any edge e of R, there is no φ' and no $W'\subseteqq W$ such that $(R-e, \varphi', f, W')$ is an admissible quadruple;

2) there is no couple φ'', $W''\subsetneqq W$ such that (R, φ'', f, W'') is an admissible quadruple;

3) if $x\in V(R)$ has degree 2 then $x\in f(G+H)$.

Put $L=L(G)\cap L(H)$. Evidently, for every graph $B\in B(L)$ there exist φ, f, W such that the admissible quadruple (B, φ, f, W) is critical.

The following lemma will finish the proof of Lemma 1.

Induction Lemma. *For every critical quadruple $Q=(R, \varphi, f, W)$, $W\neq\emptyset$, there exists a critical quadruple $Q'=(R', \varphi', f', W')$ such that $|R'|\geqq|R|$ and $W'\subsetneqq W$.*

Using the Induction Lemma, Lemma 1 may be proved as follows. For every $B\in B(L)$ there exists a critical quadruple $Q=(R, \varphi, f, W)$ such that $|R|\geqq|B|$ and $W=\emptyset$. We construct an (m, n)-crossing system from the quadruple Q by splitting every vertex $f(x)$, $x\in V(G+H)$ into $d(x)$ vertices of degree 1 where $d(x)$ is the degree of the vertex $f(x)$ in R. Since Q is critical, this (m, n)-crossing system is minimal.

Proof of the Induction Lemma. Let $Q=(R, \varphi, f, W)$ be a critical quadruple, $W\neq\emptyset$. Take a point $u\in W$. We will construct a quadruple $Q'=(R', \varphi', f', W')$ such that $W'\subseteqq W-\{u\}$ and $|R'|\geqq|R|$. Put $w=f(u)$. There are three possibilities:

a) $w\in f(G)\cap f(H)$,

b) $w\in f(G)\cap(\varphi H-f(H))$,

c) $w\in f(H)\cap(\varphi G-f(G))$.

Cases b) and c) are symmetric, consequently it suffices to treat a) and b) only. Denote by $N(x, \varphi G)$, resp. $N(x, \varphi H)$, the neighbourhood of the vertex $x\in V(R)$ in φG, resp. φH.

Case a). Let $w=f(a)=f(b)$ where $a\in V(G)$, $b\in V(H)$. Clearly, $|N(w, \varphi G)|=d_G(a)$, $|N(w, \varphi H)|=d_H(b)$, and from condition 1) in the definition of critical quadruples, $d_R(w)=|N(w, \varphi G)\cup N(w, \varphi H)|$. Next we define the admissible quadruple Q'. Let $V(R')=(V(R)-\{w\})\cup\{a', b'\}$ and defines the edges of R' by

$$e\in E(R') \quad \text{for} \quad w\notin e\in E(R),$$

$$(x, a')\in E(R') \quad \text{for} \quad x\in N(w, \varphi G),$$

$$(x, b')\in E(R') \quad \text{for} \quad x\in N(w, \varphi H).$$

The vertex mapping f' is defined by $f'(x)=f(x)$ for $x\neq a, b, f'(a)=a', f'(b)=b'$. Now we define the morphism φ'. If an edge e is not adjacent to a or b in $G+H$, put $\varphi'(e)=\varphi(e)$. If $e=(a, v)$, resp. $e=(b, v)$, $\varphi(e)=(w, x_1, \dots, x_k, f(v))$, put $\varphi'(e)=(a', x_1, \dots, x_k, f(v))$, resp. $\varphi'(e)=(b', x_1, \dots, x_k, f(v))$.

Evidently, $W\varphi'=W\varphi-\{a, b\}$. We verify that the quadruple Q' satisfies condition 1) in the definition of critical quadruples. By way of contradiction let us suppose that there is an edge $e_0\in E(R')$ and a morphism ψ' such that $(R'-e_0, \psi', f', W\psi')$ is an admissible quadruple with $W\psi'\subseteq W\varphi'$. Since $d_{R'}(a')=d_G(a)$ and $d_{R'}(b')=d_G(b)$, neither a' nor b' is adjacent to e_0. If e is an edge of G, resp. H, which is not adjacent to the vertex a, resp. b, then $b'\notin\psi'(e)$, resp. $a'\notin\psi'(e)$, because $a, b\notin W\psi'$. Thus, we can define an admissible quadruple $(R-e_0, \psi, f, W\psi)$ where the morphism ψ is defined from ψ' by the reverse procedure to the one we used to obtain φ' from φ. Clearly, $W\psi=W\psi'\cup\{a, b\}\subseteq W\varphi$. This contradicts the fact that Q is critical. Condition 3) is obvious. If Q does not satisfy condition 2), it is sufficient to replace φ' by a suitable φ'' and W' by a smaller set W''.

Case b) will be divided into three subcases b_0), b_1), b_2) where b_i) means $|N(w, \varphi G)\cap N(w, \varphi H)|=i$. Let $w=f(a)$, $a\in V(G)$.

Cases b_0) and b_1) will be considered together. Denote by x_0 an arbitrary element of $N(w, \varphi G)$ in the case b_0) and the only element of $N(w, \varphi G)\cap N(w, \varphi H)$ in the case b_1).

1. Put $V(R')=V(R)\cup\{a'\}$. Define the edges of R' by

$$e\in E(R') \quad \text{for} \quad w\notin e\in E(R),$$

$$(x, a')\in E(R') \quad \text{for} \quad x\in N(w, \varphi G)\cup\{w\}, \quad x\neq x_0,$$

$$(x, w)\in E(R') \quad \text{for} \quad x\in N(w, \varphi H)\cup\{x_0\}.$$

2. The mapping f' is defined by $f'(a)=a'$ and $f'(v)=f(v)$ for $v\neq a$.

3. The morphism φ' is defined by

$$\varphi'(e) = \varphi(e) \quad \text{for} \quad a \notin e \in E(G+H),$$

$$\varphi'(e) = (a', x_1, \ldots, x_k) \quad \text{for} \quad a \in e, \quad \varphi(e) = (w, x_1, \ldots, x_k), \quad x_1 \neq x_0,$$

$$\varphi'(e) = (a', w, x_0, x_1, \ldots, x_k) \quad \text{for the only } e \text{ with} \quad \varphi(e) = (w, x_0, x_1, \ldots, x_k).$$

4. $W' = W - \{a\}$.

Case b_2). Denote by x_1, x_2 the elements of $N(w, \varphi G) \cap N(w, \varphi H)$.

1. Put $V(R') = (V(R) \cup \{a'\}) - \{w\}$. Define the edges of R' by

$$e \in E(R') \quad \text{for} \quad w \notin e \in E(R),$$

$$(x, a') \in E(R') \quad \text{for} \quad x \in N(w, \varphi G),$$

$$(x_1, x_2) \in E(R').$$

2. Put $f'(a) = a'$ and $f'(v) = f(v)$ for $v \neq a$.

3. Put $\varphi'(e) = \varphi(e)$ for $\varphi(e)$ not containing w,

$$\varphi'(e) = (a', y_1, \ldots, y_k) \quad \text{for} \quad \varphi(e) = (w, y_1, \ldots, y_k),$$

$\varphi'(e) = (\ldots, x_1, x_2, \ldots)$ for the only $e \in E(H)$ for which $\varphi(e) = (\ldots, x_1, w, x_2, \ldots)$ contains w.

4. $W' = W - \{a\}$.

The proof of case b) goes like the proof of case a) (but in case b_2) it is also necessary to use condition 2) in the definition of critical quadruples), and we omit it. (See Fig. 2.)

Lemma 2 below implies immediately the proof of the implication 3)$\Rightarrow$4) and completes the proof of the Theorem.

Lemma 2. *Let S be a minimal (m, m)-crossing system. Then there exists a graph B from the base of the class $L(K_6^1 + \ldots + K_6^m) \cap L(K_{2,6}^1 + \ldots + K_{2,6}^m)$ such that the number of vertices of B is greater than the number of vertices of S. (K_6^i, $K_{2,6}^i$ denote the i-th copy of K_6, $K_{2,6}$, respectively.)*

Proof. Let S be a minimal (m, m)-crossing system formed by two disjoint systems of paths C, D where $C = (c_1, \ldots, c_m)$, $D = (d_1, \ldots, d_m)$. First, take the disjoint union of m copies of the graph K_6 (denote the *i-th* copy by K_6^i). The vertices of

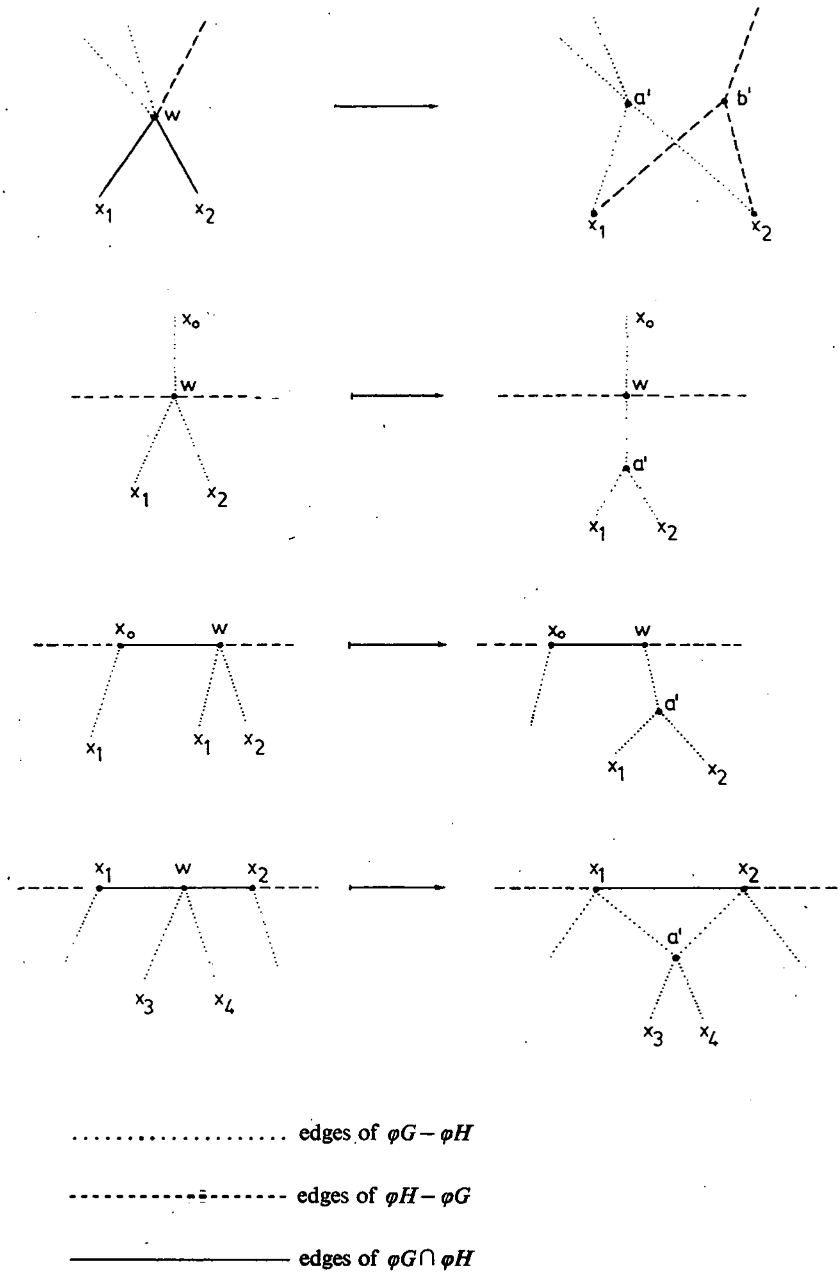
w
x_1
x_2
a'
b'
x_0
x_3
x_4
edges of $\varphi G-\varphi H$
edges of $\varphi H-\varphi G$
edges of $\varphi G\cap\varphi H$

Fig. 2

K_6^i are denoted by $c_i^1, c_i^2, d_i^1, d_i^2, u_i^1, u_i^2$. We shall construct the graph B from $\sum K_6$ in two steps:

a) we construct a subdivision of $\sum K_6$,

b) we add further edges.

Put a new vertex u_i^3 on every edge (c_i^1, c_i^2). Identify the vertices c_i^1, c_i^2 with the endpoints of the path c_i. Subdivide the edge (d_i^1, d_i^2) by the number of vertices of the path d_i and identify this subdivision with the path d_i. Now, all vertices of the crossing system S are identified with some vertices of the graph B. Hence we may assume $V(S) \subseteqq V(B)$. Add to B all edges $(x, y) \in E(\mathrm{gr}(C, D))$. This completes the construction of B.

The graph B evidently contains the subdivision of the graph $G = \sum K_6$. We shall show that it contains the subdivision of $H = \sum K_{2,6}$, too. The graph $K_{2,6}$ is formed by six paths of length 2 which have common endpoints. The subdivision of $K_{2,6}^i$ is in the graph B formed by the paths c_i^1, u_i^j, c_i^2, $j = 1, 2, 3$, c_i^1, d_i^j, c_i^2, $j = 1, 2$ and c_i^1, c_i, c_i^2 (c_i is the path of the system C).

We shall prove that the graph B does not contain other subdivisions of $\sum K_6$ and $\sum K_{2,6}$ than those described above. The vertices $u_i^1, u_i^2, c_i^1, c_i^2, d_i^1, d_i^2$, $i = 1, \ldots, m$ are the only vertices of B of degree $\geqq 5$. Hence, the only subdivision of $\sum K_6$ in B, possibly with the exception of edges (d_i^1, d_i^2), is that described above. Since the vertices c_i^1, c_i^2, $i = 1, \ldots, m$, are the only ones of degree 6 in B, the vertices of degree 6 in $K_{2,6}^i$ must be put on them. Further, 5 vertices of $K_{2,6}^i$ must be put on vertices $d_i^1, d_i^2, u_i^1, u_i^2, u_i^3$. Thus subdivisions of edges (d_i^1, d_i^2) and the remaining paths between c_i^1, c_i^2 in $K_{2,6}^i$ correspond to the paths in the crossing system S. The minimality of B follows from the minimality of S.

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